

Presented at the IUGG95 Assembly, Boulder, Colorado, July 2-14, 1995.

# Mixed integer programming for the resolution of GPS carrier phase ambiguities

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## Abstract

*This arXiv upload is to clarify that the now well-known sorted QR MIMO decoder was first presented in the 1995 IUGG General Assembly. We clearly go much further in the sense that we directly incorporated reduction into this one step, non-exact suboptimal integer solution. Except for these first few lines up to this point, this paper is an unaltered version of the paper presented at the IUGG1995 Assembly in Boulder.*

*The Ambiguity resolution of GPS carrier phase observables is crucial in high precision geodetic positioning and navigation applications. It consists of two aspects: estimating the integer ambiguities in the mixed integer observation model and examining whether they are sufficiently accurate to be fixed as known nonrandom integers. We shall discuss the first point in this paper from the point of view of integer programming. A one-step nonexact approach is proposed by employing minimum diagonal pivoting Gaussian decompositions, which may be thought of as an improvement of the simple rounding-off method, since the weights and correlations of the floating-estimated ambiguities are fully taken into account. The second approach is to reformulate the mixed integer least squares problem into the standard **0-1** linear integer programming model, which can then be solved by using, for instance, the practically robust and efficient simplex algorithm for linear integer programming. It is exact, if proper bounds for the ambiguities are given. Theoretical results on decorrelation by unimodular transformation are given in the form of a theorem.*

## 1 Introduction

Three types of observables may be derived from tracking GPS satellites: pseudorange (code) measurements, raw Doppler shifts (or equivalently range rates) and carrier phases. They are used at different levels of accuracy for different purposes of applications (see e.g Wells et al. 1986; Leick 1990; Hofmann-Wellenhof et al. 1992; Seeber 1993; Melbourne 1985). The carrier phase measurements, together with the accurate code observables (if available), have been dominating in high precision geodetic positioning and navigation applications. The mathematical model can symbolically be written below

$$\mathbf{R} = \mathbf{f}_R(\mathbf{X}) + \mathbf{B}_R\boldsymbol{\lambda} + \varepsilon_R \quad (1a)$$

$$\boldsymbol{\Phi} = \mathbf{f}_\Phi(\mathbf{X}) + \mathbf{B}_\Phi\boldsymbol{\lambda} + \mathbf{B}_Z\mathbf{Z} + \varepsilon_\Phi. \quad (1b)$$

Here  $\mathbf{R}$  and  $\boldsymbol{\Phi}$  are respectively the observables of pseudoranges and carrier phases,  $\varepsilon_R$  and  $\varepsilon_\Phi$  are the random errors of the observables,  $\mathbf{X}$  is the coordinate vector to be estimated, and  $\mathbf{f}_R(\cdot)$  and  $\mathbf{f}_\Phi(\cdot)$  are nonlinear functionals of  $\mathbf{X}$ .  $\mathbf{B}_R$ ,  $\mathbf{B}_\Phi$  and  $\mathbf{B}_Z$  are the coefficient matrices.  $\boldsymbol{\lambda}$  is the vector of nuisance parameters such as the synchronization errors of receiver and satellite clocks and ionospheric corrections. If overparametrization occurs to  $\boldsymbol{\lambda}$ , it is generally not estimable (Wells et

al. 1987). Thus we shall assume that proper reparametrization has been made by, for instance, choosing proper datum parameters (Wells et al. 1987) or using differencing and nuisance parameter elimination techniques (see e.g. Goad 1985; Schaffrin & Grafarend 1986), to ensure that the remaining nuisance parameters are estimable.  $\mathbf{Z}$  is the vector of integral ambiguities inhered in the carrier phase observables.

Accurate and reliable resolution of the integral ambiguity vector has been playing a crucial role in high precision positioning. There are currently many approximate proposals available to resolve  $\mathbf{Z}$ . They may be treated in two categories: simple (sequential) rounding-off of a real number to its nearest integer with and/or without using constraint criteria (Blewitt 1989; Talbot 1991; Hwang 1991; Seeber 1993; Hofmann-Wellenhof et al. 1992), and searching methods by employing the information on the prior statistics and geometry (nonlinear functionals and design matrices) of the observables (Counselman et al. 1981; Remondi 1990, 1991; Frei & Beutler 1990; Mader 1990; Mervart et al. 1994). Betti, Crespi & Sansò (1993) recently proposed a Bayesian approach to resolution of ambiguity. Chen & Lachapelle (1994) proposed a fast ambiguity search filtering approach to reducing the number of possible candidates in the searching area. It may be worth noting that the fast rapid ambiguity resolution method proposed by Frei & Beutler seems to have enjoyed its wide approval. A key element of the method is the use of some formal statistics to pick up a solution. It may be proved that the statistic used for selecting the candidates of ambiguities is not mathematically rigorous, since the ambiguity-free and ambiguity-fixed solution vectors are both derived by using the same set of carrier phase observations. The method seems quite successful in practice, however.

Recent progress in resolving the integral ambiguity vector has been made by Teunissen (1994). His approach consists of three steps: (1) decorrelation of the floating-estimated ambiguities by Gaussian transformation, which may be said to characterize the novelty of the new approach, (2) searching for the solution to the transformed integer least squares problem within a superellipsoid corresponding to a certain level of confidence, and (3) back-substituting the solution just derived for the ambiguity vector in the original model. The success of the approach will depend, to a great extent, on the first two steps. Testing results of the approach can be found in Teunissen (1994) and de Jonge & Tiberius (1994). Decorrelation techniques may be also well suited to explain an important finding by Melbourne (1985), that the widelane ambiguity is easier to solve, based on the *one epoch* dual frequency carrier phase and code-derived pseudorange model.

The purpose of this paper is to further study the GPS ambiguity resolution as a mixed integer least squares (LS) mathematical programming problem. Unimodular integer transformation is used to statistically decorrelate the floating-estimated ambiguities, which summarizes the first two conditions of transformation proposed by Teunissen (1994). Two methods for solving the transformed integer LS problem are then proposed. The first one is to decompose the transformed positive definite matrix into a lower and an upper triangle by choosing the minimum diagonal elements. In this way, we are sure that a wrongly selected ambiguity will be penalized. No iterations are required, thus it should improve the sum of square of the residuals derived by rounding the transformed real values to their nearest integers. The second one is to reformulate the transformed integer LS problem to a quadratic **0-1** nonlinear programming, and then further to a **0-1** linear integer programming. Thus simplex algorithms can be employed to efficiently solve the linear integer programming problem, with which one need not test every point in the feasible solution set.

## 2 Integer and mixed integer least squares models

In the application of the GPS system to high precision positioning and navigation, the GPS satellites have been treated as space targets with known positions, unless the determination of the satellite orbits is of interest. In this paper, we assume that the coordinates of the satellites are given, which can be computed, for instance, from the (precision) ephemerides. Furthermore, given a set of approximate coordinates of the stations, we can linearize the observation equations (1a) and (1b)

as

$$\mathbf{y}_R = \mathbf{A}_R \Delta \mathbf{X} + \mathbf{B}_R \Delta \boldsymbol{\lambda} + \varepsilon_R \quad (2a)$$

$$\mathbf{y}_\Phi = \mathbf{A}_\Phi \Delta \mathbf{X} + \mathbf{B}_\Phi \Delta \boldsymbol{\lambda} + \mathbf{B}_Z \Delta \mathbf{Z} + \varepsilon_\Phi \quad (2b)$$

where

$$\mathbf{y}_R = \mathbf{R} - \mathbf{f}_R(\mathbf{X}_0) - \mathbf{B}_R \boldsymbol{\lambda}_0 \quad (3a)$$

$$\mathbf{y}_\Phi = \boldsymbol{\Phi} - \mathbf{f}_\Phi(\mathbf{X}_0) - \mathbf{B}_\Phi \boldsymbol{\lambda}_0 - \mathbf{B}_Z \mathbf{Z}_0 \quad (3b)$$

$$\Delta \mathbf{X} = \mathbf{X} - \mathbf{X}_0; \quad \Delta \boldsymbol{\lambda} = \boldsymbol{\lambda} - \boldsymbol{\lambda}_0 \quad (3c)$$

$$\Delta \mathbf{Z} = \mathbf{Z} - \mathbf{Z}_0. \quad (3d)$$

$\mathbf{X}_0$  and  $\boldsymbol{\lambda}_0$  are the approximate values of  $\mathbf{X}$  and  $\boldsymbol{\lambda}$ , respectively.  $\mathbf{Z}_0$  are integer approximate values of  $\mathbf{Z}$ , and thus  $\Delta \mathbf{Z}$  remain integral.

Rewriting the linearized observation equations (2a) and (2b) in matrix form, together with the statistical information on the observables, we have

$$\begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_\Phi \end{bmatrix} = \begin{bmatrix} \mathbf{A}_R \\ \mathbf{A}_\Phi \end{bmatrix} \Delta \mathbf{X} + \begin{bmatrix} \mathbf{B}_R \\ \mathbf{B}_\Phi \end{bmatrix} \Delta \boldsymbol{\lambda} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_Z \end{bmatrix} \Delta \mathbf{Z} + \begin{bmatrix} \varepsilon_R \\ \varepsilon_\Phi \end{bmatrix} \quad (4a)$$

$$D \begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_\Phi \end{bmatrix} = \begin{bmatrix} \mathbf{P}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\Phi \end{bmatrix}^{-1} \sigma^2. \quad (4b)$$

Here  $\mathbf{P}_R$  and  $\mathbf{P}_\Phi$  are respectively the weight matrices of the observables  $\mathbf{y}_R$  and  $\mathbf{y}_\Phi$ ,  $\sigma^2$  is the scalar variance component.

Since the main interest of this paper is to discuss the mixed integer LS problem, we do not need to discriminate between the position unknowns  $\mathbf{X}$  and the nuisance parameters  $\boldsymbol{\lambda}$ . Without loss of generality, therefore, we can simplify the model (4) as the following standard mixed real-integer (or simply integer in the rest of the paper) observation equations,

$$\mathbf{y} = \mathbf{A}\boldsymbol{\beta} + \mathbf{B}\mathbf{z} + \varepsilon \quad (5a)$$

$$D(\mathbf{y}) = \mathbf{P}^{-1} \sigma^2 \quad (5b)$$

where

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_\Phi \end{bmatrix}; \quad \varepsilon = \begin{bmatrix} \varepsilon_R \\ \varepsilon_\Phi \end{bmatrix} \\ \mathbf{A} &= \begin{bmatrix} \mathbf{A}_R & \mathbf{B}_R \\ \mathbf{A}_\Phi & \mathbf{B}_\Phi \end{bmatrix}; \quad \boldsymbol{\beta} = \begin{bmatrix} \Delta \mathbf{X} \\ \Delta \boldsymbol{\lambda} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_Z \end{bmatrix}; \quad \mathbf{z} = \Delta \mathbf{Z} \\ \mathbf{P} &= \begin{bmatrix} \mathbf{P}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\Phi \end{bmatrix}. \end{aligned}$$

The matrices  $\mathbf{A}$  and  $\mathbf{B}$  are full of column rank, respectively.

Applying the least squares criterion to (5), we have

$$\min : \quad F = (\mathbf{y} - \mathbf{A}\boldsymbol{\beta} - \mathbf{B}\mathbf{z})^T \mathbf{P} (\mathbf{y} - \mathbf{A}\boldsymbol{\beta} - \mathbf{B}\mathbf{z}), \quad (6)$$

which is the mixed integer LS problem. (6) was also called the constrained LS problem by Teunissen (1994). Since the variables  $\mathbf{z}$  are discrete, we cannot use the conventional method by differentiating the objective function  $F$  with respect to the variables  $\boldsymbol{\beta}$  and  $\mathbf{z}$  in order to form the normal equation

and then solve for them. Instead, however, we differentiate  $F$  with respect to  $\beta$  and let it equal zero, leading to

$$\frac{\partial F}{\partial \beta} = -2\mathbf{A}^T \mathbf{P}(\mathbf{y} - \mathbf{A}\beta - \mathbf{B}\mathbf{z}) = \mathbf{0}$$

or

$$\mathbf{A}^T \mathbf{P} \mathbf{A} \beta = \mathbf{A}^T \mathbf{P}(\mathbf{y} - \mathbf{B}\mathbf{z}).$$

Hence

$$\beta = (\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1}(\mathbf{y} - \mathbf{B}\mathbf{z}). \quad (7)$$

Substituting (7) into (5) and rearranging it yield

$$\mathbf{y}_1 = \mathbf{Q} \mathbf{P} \mathbf{B} \mathbf{z} + \varepsilon_1 \quad (8a)$$

$$D[\mathbf{y}_1] = [\mathbf{P}^{-1} - \mathbf{A}(\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1} \mathbf{A}^T] \sigma^2 = \mathbf{Q} \sigma^2 \quad (8b)$$

where

$$\begin{aligned} \mathbf{y}_1 &= [\mathbf{I} - \mathbf{A}(\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{P}] \mathbf{y} = \mathbf{Q} \mathbf{P} \mathbf{y} \\ \mathbf{Q} &= \mathbf{P}^{-1} - \mathbf{A}(\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1} \mathbf{A}^T. \end{aligned}$$

Applying the LS method to (8), we have

$$\begin{aligned} \min : F_1 &= (\mathbf{y}_1 - \mathbf{Q} \mathbf{P} \mathbf{B} \mathbf{z})^T \mathbf{Q}^{-1} (\mathbf{y}_1 - \mathbf{Q} \mathbf{P} \mathbf{B} \mathbf{z}) \\ &= (\mathbf{y} - \mathbf{B} \mathbf{z})^T \mathbf{P} \mathbf{Q} \mathbf{Q}^{-1} \mathbf{Q} \mathbf{P} (\mathbf{y} - \mathbf{B} \mathbf{z}) \\ &= (\mathbf{y} - \mathbf{B} \mathbf{z})^T \mathbf{P} \mathbf{Q} \mathbf{P} (\mathbf{y} - \mathbf{B} \mathbf{z}) \\ &= \mathbf{y}^T \mathbf{P} \mathbf{Q} \mathbf{P} \mathbf{y} - 2\mathbf{y}^T \mathbf{P} \mathbf{Q} \mathbf{P} \mathbf{B} \mathbf{z} + \mathbf{z}^T \mathbf{B}^T \mathbf{P} \mathbf{Q} \mathbf{P} \mathbf{B} \mathbf{z}. \end{aligned} \quad (9)$$

The objective function  $F_1$  can further be rewritten as

$$\min : F_1 = (\mathbf{z} - \hat{\mathbf{z}})^T \mathbf{H} (\mathbf{z} - \hat{\mathbf{z}}) + \mathbf{y}^T \mathbf{P} \mathbf{Q} [\mathbf{Q}^{-1} - \mathbf{P} \mathbf{B} \mathbf{H}^{-1} \mathbf{B}^T \mathbf{P}] \mathbf{Q} \mathbf{P} \mathbf{y} \quad (10)$$

where

$$\begin{aligned} \hat{\mathbf{z}} &= \mathbf{H}^{-1} \mathbf{B}^T \mathbf{P} \mathbf{Q} \mathbf{P} \mathbf{y} \\ \mathbf{H} &= (\mathbf{B}^T \mathbf{P} \mathbf{Q} \mathbf{P} \mathbf{B}). \end{aligned}$$

Here  $\hat{\mathbf{z}}$  can readily be proved to be the floating LS estimate of the ambiguity vector  $\Delta \mathbf{Z}$  with covariance matrix  $\mathbf{H}^{-1} \sigma^2$ . Since  $\mathbf{y}^T \mathbf{P} \mathbf{Q} [\mathbf{Q}^{-1} - \mathbf{P} \mathbf{B} \mathbf{H}^{-1} \mathbf{B}^T \mathbf{P}] \mathbf{Q} \mathbf{P} \mathbf{y}$  is constant, the objective function (10) is equivalent to (Teunissen 1994; de Jonge & Tiberius 1994)

$$\min : F_2 = (\mathbf{z} - \hat{\mathbf{z}})^T \mathbf{H} (\mathbf{z} - \hat{\mathbf{z}}), \quad (11)$$

which is the standard integer LS problem.

It is now clear that the solution to the original mixed integer LS problem (6) depends solely on that of the standard integer LS problem (11). Denote the integer solution of  $\mathbf{z}$  to (11) by  $\hat{\mathbf{z}}^{IN}$ . Substituting it into (7), we can then obtain the LS estimates of the real parameters  $\beta$  without much effort.

### 3 Unimodular transformation

In resolution of GPS carrier phase ambiguities, one of the most difficult points is to handle strong correlation of the matrix  $\mathbf{H}$ . Searching for an acceptable (and/or hopefully optimal) solution of  $\mathbf{z}$  is arduous, if it is solely based on the strong correlation matrix  $\mathbf{H}$ , since testing a large number of combinations would have to be done. Roughly speaking, the total number of combinations required is computed by  $\prod n_i$ , where  $n_i$  is the number of integer points on an interval of line for the  $i$ th ambiguity, centred at the  $\hat{z}_i$  and corresponding to a significance level (see e.g. Frei & Beutler 1990). However, if the matrix  $\mathbf{H}$  is diagonal, one can simply round the floating values  $\hat{\mathbf{z}}$  off to the nearest integers, which are the integer solution of  $\mathbf{z}$ . Therefore, an idea would emerge naturally, that one works with a decorrelated weight matrix instead of  $\mathbf{H}$ .

Such a technique was proposed recently by Teunissen (1994) (see also de Jonge & Tiberius 1994). His basic idea is to transform the “observables”  $\hat{\mathbf{z}}$  by  $\mathbf{G}$  into the new ones  $\hat{\mathbf{z}}_1 = (\mathbf{G}^T \hat{\mathbf{z}})$ , and then work with the LS integer problem

$$\min : F_3 = (\mathbf{z}_1 - \hat{\mathbf{z}}_1)^T \mathbf{H}_1 (\mathbf{z}_1 - \hat{\mathbf{z}}_1). \quad (12)$$

Here  $\mathbf{H} = \mathbf{G} \mathbf{H}_1 \mathbf{G}^T$ . The transformation matrix  $\mathbf{G}$  has to satisfy the following three conditions: (1) integer elements; (2) volume preservation; and (3) decorrelation of  $\mathbf{H}$  into  $\mathbf{H}_1$ . More details can be found in Teunissen (1994).

Before proceeding, we shall define the unimodular matrix (see e.g. Nemhauser & Wolsey 1988).

**Definition 1.** A square matrix  $\mathbf{G}$  is said to be *unimodular* if it is integral and if the absolute value of its determinant is equal to unity, i.e.  $|\det(\mathbf{G})| = 1$ .

The inverse of a unimodular matrix is also unimodular, since  $|\det(\mathbf{G}^{-1})| = 1/|\det(\mathbf{G})| = 1$ , and because

$$\mathbf{G}^{-1} = \bar{\mathbf{G}} / \det(\mathbf{G}) = \pm \bar{\mathbf{G}}.$$

Here  $\bar{\mathbf{G}}$  is the *adjoint matrix* of  $\mathbf{G}$ , whose elements are derived only by using the operations of integer multiplication, subtraction and addition, and thus integer. The sign before  $\bar{\mathbf{G}}$  depends on the determinant of  $\mathbf{G}$ . The second property of unimodular matrices is that the product of two unimodular matrices is unimodular. It is also clear that any unimodular transformation of an integer vector is an integer vector, too.

By employing the concept of the unimodular matrix, we can summarize the first two conditions suggested by Teunissen (1994) by stating that the transformation  $\mathbf{G}$  is unimodular. It should be noted that there was a misunderstanding of Teunissen’s second condition of volume preservation. Volume preservation does not imply the preservation of the number of grid points. A simple example is that a unit circle centred at the origin has five grid points, while an ellipse of the same center with major axis 1.5 and minor axis 2/3 encloses only three grid points.

Integer Gaussian decomposition was employed by Teunissen (1994), that indeed decorrelates the matrix  $\mathbf{H}$ . What now seems to be done is to mathematically prove that we can always decorrelate the matrix  $\mathbf{H}$  by using a finite number of unimodular transformations to the extent that the correlation coefficient of any two random variables is always less than or equal to 1/2. In order to do so, we need the following lemma on the inequality of matrix determinant.

**Lemma 1:** For any positive definite matrix  $\mathbf{A}$ , the following inequality

$$\det(\mathbf{A}) \leq \prod a_{ii} \quad (13)$$

holds true. Here  $a_{ii}$  are the diagonal elements of  $\mathbf{A}$ .

**Proof.** A positive definite matrix  $\mathbf{A}$  can be written by Choleski decomposition as

$$\mathbf{A} = \mathbf{L} \mathbf{L}^T$$

where  $l_{ii} = (a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2)^{1/2} > 0$ . Thus we have

$$\begin{aligned} \det(\mathbf{A}) &= \prod l_{ii}^2 \\ &= \prod (a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2) \\ &\leq \prod a_{ii}, \end{aligned}$$

since  $\sum_{j=1}^{i-1} l_{ij}^2 \geq 0$ .  $\square$

**Theorem 1:** For any positive definite matrix  $\mathbf{A}$ , there exists a unimodular matrix  $\mathbf{G}$  such that

$$\mathbf{A} = \mathbf{G}\mathbf{H}\mathbf{G}^T. \quad (14)$$

Here  $\mathbf{H}$  is positive definite, too, and satisfies

$$|h_{ij}| \leq \frac{1}{2} \min(h_{ii}, h_{jj}) \quad \forall i, j \text{ \& } i \neq j. \quad (15)$$

**Proof.** Suppose, without loss of generality, that for any three elements  $a_{ii}$ ,  $a_{jj}$  and  $a_{ij}$  of the positive definite matrix  $\mathbf{A}$ , we have  $|a_{ij}|/\min(a_{ii}, a_{jj}) > 1/2$ . Then construct the unimodular matrix

$$\mathbf{G}_1 = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \vdots & & \ddots & \\ & & -[a_{ij}/a_{ii}]_{in} & \cdots & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \quad (16a)$$

if  $a_{ii} \leq a_{jj}$ , or

$$\mathbf{G}_1 = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & \cdots & -[a_{ij}/a_{jj}]_{in} & \\ & & & \ddots & \vdots & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \quad (16b)$$

if  $a_{jj} < a_{ii}$ . Here  $[x]_{in}$  is the operation to round the floating number  $x$  to its nearest integer.

Upon left- and right-multiplying  $\mathbf{A}$  by the unimodular matrix  $\mathbf{G}_1$  and its transpose respectively, the larger diagonal element is then reduced to

$$\max(a_{ii}, a_{jj}) - 2[a_{ij}/a_{min}]_{in}a_{ij} + a_{min}[a_{ij}/a_{min}]_{in}^2 \quad (17)$$

where  $a_{min} = \min(a_{ii}, a_{jj})$ . Repeating the same procedure to any pair of diagonal elements, we have

$$\mathbf{A}_n = \mathbf{G}_n \dots \mathbf{G}_1 \mathbf{A} \mathbf{G}_1^T \dots \mathbf{G}_n^T. \quad (18)$$

Now suppose that we cannot reach the equation (14) and the inequality (15) by employing a finite number of unimodular matrices of the form (16), then we keep applying the same procedure to  $\mathbf{A}_n$ . By expression (17), it is clearly true that the minimum diagonal element of the reduced

matrix, say  $\mathbf{A}_m$  now, has no lower bound. It means that the minimum element can be arbitrarily small, which further implies by Lemma 1 that

$$\det(\mathbf{A}_m) \leq \prod a_{ii}^m < \text{const}, \quad (19)$$

where  $a_{ii}^m$  are the diagonal elements of  $\mathbf{A}_m$ , *const* is any positive constant. Since unimodular transformation does preserve the determinant, we have  $\det(\mathbf{A}_m) = \det(\mathbf{A})$  — a finite constant, which clearly contradicts (19). Therefore, we must be able to reach the condition (15). On the other hand, all the transformation matrices involved are unimodular, their product is unimodular, too. Denoting the final reduced matrix by  $\mathbf{H}$ , which satisfies the condition (15), and the product of all the unimodular matrices by  $\mathbf{G}_t$ , we have

$$\mathbf{H} = \mathbf{G}_t \mathbf{A} \mathbf{G}_t^T \quad (20)$$

or

$$\mathbf{A} = \mathbf{G} \mathbf{H} \mathbf{G}^T. \quad (21)$$

Here  $\mathbf{G}(= \mathbf{G}_t^{-1})$  is unimodular. The proof that the matrix  $\mathbf{H}$  is positive definite is trivial.  $\square$

## 4 Two approaches to the integer LS problem

The integer LS problem is simply an integer quadratic programming issue. One can use any advanced integer programming algorithm (Parker & Rardin 1988) to solve this problem. Essentially, no bounds for the integer unknowns are required and no statistical techniques needed to reduce the number of possible candidates. More on these aspects and proper validation criteria for fixing the carrier phase ambiguities will be presented in a future paper.

Though the techniques to be presented below require no decorrelation as an assumption, and consider that the original and the transformed LS integer problems are of the same form, the following discussion will be based on the transformed model, without loss of generality. After the weight matrix of the floating-estimated ambiguity vector is decorrelated, one can either simply round the transformed floating numbers off to their nearest integers, or employ searching techniques to find the “optimal” solution within a superellipsoid under a certain level of confidence (Teunissen 1994). In what follows, we shall develop two approaches to resolve the ambiguities of the transformed integer LS problem.

### 4.1 A one-step nonexact approach by minimum diagonal pivoting Gaussian decomposition

Instead of directly applying the simple rounding-off method to (12), which ignores any correlation information on the floating-estimated ambiguities, we propose an alternative one-step approach, based on the weights and correlations of the transformed ambiguities. The basic idea is to resolve the integer ambiguities according to their weights and correlations. As long as some of ambiguities are resolved, their correlations with other unfixed floating ambiguities are employed and the next ambiguity corresponding to the large weight is to be determined.

In order to realize the above procedure, we have to decompose the positive definite matrix  $\mathbf{H}_1$  carefully. Here we employ Gaussian decomposition by selecting the minimum diagonal element. The decomposition procedure consists of the following steps:

- Selecting the minimum element among all the undecomposed diagonal elements;
- Exchanging the rows and the columns;
- Performing Gaussian decomposition;

- Replacing the square root of the decomposed element  $h'_{1(ii)}$  at the corresponding position of the factor matrix  $\mathbf{L}$ ; If the decomposition is not completed, then go to the first step. Otherwise, the decomposition is finished.

In mathematical language, we can express the matrix  $\mathbf{H}_1$  as

$$\mathbf{H}_1 = \mathbf{P}_h \mathbf{L} \mathbf{L}^T \mathbf{P}_h^T \quad (22)$$

where  $\mathbf{P}_h$  is the permutation matrix which represents the exchange of the rows and columns during the decomposition. A significant characteristic of this decomposition is to keep the diagonal elements of the lower triangular matrix  $\mathbf{L}$  in the increasing order as far as possible.

Inserting  $\mathbf{H}_1$  in (22) into (12), we have the objective function

$$\begin{aligned} \min : F_3 &= (\mathbf{z}_1 - \hat{\mathbf{z}}_1)^T \mathbf{P}_h \mathbf{L} \mathbf{L}^T \mathbf{P}_h^T (\mathbf{z}_1 - \hat{\mathbf{z}}_1) \\ &= (\mathbf{z}_2 - \hat{\mathbf{z}}_2)^T \mathbf{L} \mathbf{L}^T (\mathbf{z}_2 - \hat{\mathbf{z}}_2) \end{aligned} \quad (23)$$

where

$$\mathbf{z}_2 = \mathbf{P}_h^T \mathbf{z}_1; \quad \hat{\mathbf{z}}_2 = \mathbf{P}_h^T \hat{\mathbf{z}}_1. \quad (24)$$

Since the factor matrix  $\mathbf{L}$  is lower triangular, we can rewrite (23) as

$$\min : F_4 = \sum_{i=1}^{t_z} \left[ \sum_{j=i}^{t_z} l_{ji} (z_{2(j)} - \hat{z}_{2(j)}) \right]^2. \quad (25)$$

Here  $t_z$  is the dimension of the ambiguity vector  $\mathbf{z}$  (or  $\mathbf{z}_2$ ). The solution to the objective function  $F_2$  can now be derived by minimizing

$$\left| \sum_{j=i}^{t_z} l_{ji} (z_{2(j)} - \hat{z}_{2(j)}) \right|, \quad \forall i. \quad (26)$$

Hence the one-step nonexact integer ambiguity solution is immediate

$$\hat{z}_{2(i)}^{IN} = \left[ \frac{l_{ii} \hat{z}_{2(i)} - \sum_{j=i+1}^{t_z} l_{ji} (\hat{z}_{2(j)}^{IN} - \hat{z}_{2(j)})}{l_{ii}} \right]_{in} \quad (27)$$

for all  $i$ .

By back substituting the integer solution  $\hat{\mathbf{z}}_2^{IN} = (\hat{z}_{2(1)}^{IN}, \hat{z}_{2(2)}^{IN}, \dots, \hat{z}_{2(t_z)}^{IN})^T$ , we have the final solution of the integer ambiguities  $\mathbf{z}$ , which is denoted by  $\hat{\mathbf{z}}^{IN}$ ,

$$\hat{\mathbf{z}}^{IN} = \mathbf{G}^{-T} \mathbf{P}_h \hat{\mathbf{z}}_2^{IN}. \quad (28)$$

## 4.2 0-1 quadratic integer programming

An obvious aim of applying the decorrelation technique to the original integer LS problem is the alleviation of the computational burden for finding the optimal ambiguity solution. When it is translated into the case of searching techniques, we expect that the total number of candidate grid points to be tested should be significantly reduced. Suppose that for the transformed integer LS problem (12) ( $\mathbf{H}_1$  satisfies the conditions of Theorem 1), we have to search for the optimal integer ambiguity resolution within the hard bounds

$$m_i^0 \leq z_{1(i)} \leq m_i^1, \quad \forall i \quad (29)$$

or in another form,

$$z_{1(i)} \in [m_{1i}(=m_i^0), m_{2i}, \dots, m_{1s_i}(=m_i^1)]. \quad (30)$$

Here  $z_{1(i)}$  is the  $i$ th integer component of the integer vector  $\mathbf{z}_1$ ,  $m_{1i}$ ,  $m_{2i}$ , ..., and  $m_{1s_i}$  are the contiguous integers — the candidate points of  $z_{1(i)}$  with the lower integer bound  $m_i^0$  and the upper integer bound  $m_i^1$ . Thus our mixed integer LS problem has been reduced to a quadratic integer programming problem with simple integer constraints.

In what follows we shall further reformulate it by a **0-1** quadratic integer programming model. It has been shown by Parker & Rardin (1988) that the integer variable  $z_{1(i)}$  can be represented with  $r_i$  **0-1** variables, *i.e.*

$$z_{1(i)} = m_i^0 + \sum_{j=0}^{r_i-1} 2^j b_{i(j)}, \quad \forall i \quad (31)$$

where  $b_{i(j)}$  are **0-1** integer (binary) variables,  $r_i = [\log_2(m_i^1 - m_i^0)]_s + 1$ , and  $[\cdot]_s$  stands for the integer not larger than the positive number in brackets.

Rewriting all the integer variables  $z_{1(i)}$  in matrix form, we have

$$\mathbf{z}_1 = \mathbf{m}^0 + \mathbf{A}_1 \mathbf{b} \quad (32)$$

where the matrix  $\mathbf{A}_1$  is integral with elements  $2^k$ ,

$$\begin{aligned} \mathbf{m}^0 &= (m_1^0, m_2^0, \dots, m_{t_z}^0)^T \\ \mathbf{b} &= (\mathbf{b}_1^T, \mathbf{b}_2^T, \dots, \mathbf{b}_{t_z}^T)^T \\ \mathbf{b}_i &= (b_{i(0)}, b_{i(1)}, \dots, b_{i(r_i-1)})^T. \end{aligned}$$

Furtheron, inserting (32) into the objective function (12) yields

$$\min : F_3 = (\mathbf{A}_1 \mathbf{b} + \mathbf{m}^0 - \hat{\mathbf{z}}_1)^T \mathbf{H}_1 (\mathbf{A}_1 \mathbf{b} + \mathbf{m}^0 - \hat{\mathbf{z}}_1) \quad (33)$$

subject to  $b_k = 0$  or  $1$  for all  $k$ .

The objective function (33) is equivalent to

$$\begin{aligned} \min : F_3 &= (\mathbf{m}^0 - \hat{\mathbf{z}}_1)^T \mathbf{H}_1 (\mathbf{m}^0 - \hat{\mathbf{z}}_1) + 2(\mathbf{m}^0 - \hat{\mathbf{z}}_1)^T \mathbf{H}_1 \mathbf{A}_1 \mathbf{b} \\ &\quad + \mathbf{b}^T \mathbf{A}_1^T \mathbf{H}_1 \mathbf{A}_1 \mathbf{b}. \end{aligned} \quad (34)$$

### 4.3 0-1 linear integer programming

In this subsection, we shall further reformulate the **0-1** quadratic programming (34) into a **0-1** linear integer programming problem by using the linearization technique. The basic idea of the linearization technique is to introduce a new variable to replace the nonzero quadratic term  $b_i b_j$ . Thus the **0-1** quadratic programming problem becomes linear. Since the new variables are obviously binary, all the variables in the linear programming model to be reformulated below are binary, too.

Denoting

$$v_k = b_i b_j, \quad k = (i-1)i/2 + j, \quad i \geq j$$

and taking the following relations

$$b_i^2 = b_i$$

into account, we have

$$\min : F_4 = (\mathbf{m}^0 - \hat{\mathbf{z}}_1)^T \mathbf{H}_1 (\mathbf{m}^0 - \hat{\mathbf{z}}_1) + \sum_{i=1}^{t_v} c_i v_i \quad (35a)$$

subject to the following constraints,

$$v_i = 0 \vee 1 \quad (35b)$$

$$v_k \geq v_{ki} + v_{kj} - 1 \quad (35c)$$

$$v_k \leq v_{ki} \quad (35d)$$

$$v_k \leq v_{kj} \quad (35e)$$

$$ki = i(i+1)/2; \quad kj = j(j+1)/2.$$

Here  $t_v$  is the dimension of the **0-1** integer vector

$$\mathbf{v} = (v_1, v_2, \dots, v_{t_v})^T.$$

Since the first term in the objective function (35a) is constant, it is equivalent to

$$\min : F_4 = \sum_{i=1}^{t_v} c_i v_i \quad (36a)$$

subject to the constraints (35b ~ e). (36) is obviously of the standard form of the **0-1** linear integer programming. It can be solved by using any standard algorithms for **0-1** linear programming (Pardalos & Li 1993; Nemhauser & Wolsey 1988; Parker & Rardin 1988; The People University of China 1987). However, the algorithm aspects for the program (36) will not be discussed here.

## 5 Concluding remarks

GPS carrier phase and pseudorange observables are essentially a nonlinear mixed integer observation model. If the GPS satellites are treated as space known targets, the model is regular. Given a set of approximate values of the unknown parameters such as the position coordinates and integer ambiguities, the nonlinear model is linearized. Estimating the parameters in the linearized mixed integer model is equivalent to solving a mixed integer LS problem (if the LS principle is employed), which can be further reduced into a standard integer LS programming.

It has been recognized that one of the difficulties in correctly estimating the integer ambiguities is due to the correlations of the floating-estimated ambiguities. A decorrelation technique has been proposed by Teunissen (1994), based on Gaussian decomposition. We have further proved mathematically that there exists a unimodular matrix such that (14) and (15) hold true, which may be thought of as a theoretical summary (and extension) of some of the results in Teunissen (1994).

Two approaches are then proposed to solve the standard linear integer LS problem (12) from the point of view of integer programming theory. The first approach is to Gauss-decompose the matrix  $\mathbf{H}_1$  by selecting the minimum diagonal elements. In other words, we are estimating the integer ambiguities according to the magnitudes of the weights of the floating-estimated ambiguities and their correlations (as far as possible). It may be thought to be an improvement of the simple rounding-off method. No iterations are required. It should be noted, however, that this method is one-step nonexact. The extent of approximation should be further investigated. The second approach is to reformulate the mixed integer LS problem into a **0-1** linear integer programming model. Thus any standard algorithms for linear integer programming problems can be employed. The method will result in the exact integer solution of the ambiguities to the original mixed integer problem, if proper bounds for the integer unknowns in the transformed model (12) are given. Testing of the techniques with real data is under way.

**Acknowledgements:** Much research for this paper was conceived and done, while PX was a research fellow of *Alexander von Humboldt* foundation at Stuttgart University with Prof. Dr. Erik Grafarend as his host. The support from the foundation and the very friendship of his host are most appreciated.

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